# This paper

### Addresses that MF analysis has been limited to the independent analysis of a single image

### Proposes a joint Bayesian model and associated estimation procedure for MF parameter of multivariate images

# Key contributions

### **A novel Fourier domain statistical model f**or a single image is developed and permits the use of a likelihood that is separable in the MF parameters of interest via a suitable reparametrization and data augmentation

### **A joint Bayesian model for multivariate images i**s formulated in which prior models based on gamma Markov Random Fields encode the assumption of smooth evolution of MF parameters between the image components

### **An efficient estimation procedure** that can handle a large number of data components

# Texture

#### 79--Statistical and structural approaches to texture

# Texture model

## ||Scale invariant process||

#### 07-PAMI-Infinitely divisible cascades to model the statistics of natural images

#### 09-ICIP-Wavelet leader multifractal analysis for texture classification

#### 10-CVPR-A new texture descriptor using multifractal analysis in multi-orientation wavelet pyramid (**Xu**)

### Motivating the use of **random fractals**, **scale invariance** or **self-similarity**

## ||Degree of point-wise singular behavior or regularity of the image amplitudes||

#### 1974--Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier

#### 03--Multifractal processes

## ||Densely interwoven sets of singularities of different regularity strengths||

### Measured by Holder exponent

#### 03--Multifractal processes

#### 04--Wavelet techniques in multifractal analysis

## ||Multifractal spectrum||

#### 92-CVPR-Multifractals, texture, and image analysis

#### 03--A wavelet-based method for multifractal image analysis: from theoretical concepts to experimental applications

### Provides a global characterization of the texture via the so-called multifractal spectrum

### Collecting the Hausdorff dimensions of the sets of positions that share the same regularity exponent

# the multifractality or intermittency parameter

### This paper is focused on one of the central paramenters in MFA,

### Related to the width of the multifractal spectrum (hence, to the degree of fluctuations of regularity exponents in the image)

### Enables discrimination between the two major classes of scale invariance processes

##### Self-similar processes (Additive construction based)

c2 = 0

##### Multifractal multiplicative cascade (multiplicative construction based) (MMC)

is strictly negative

12-SSP-Multifractal analysis of self-similar processes

# Main concepts of multifractal analysis

#### 03--Multifractal processes

#### 04--Wavelet techniques in multifractal analysis

#### 15--Irregularities and scaling in signal and image processing: multifractal analysis

# MFA application

### Texture classification

###### 09-ICIP-Wavelet leader multifractal analysis for texture classification

###### 10-CVPR-A new texture descriptor using multifractal analysis in multi-orientation wavelet pyramid (Xu)

### Biomedical imaging

###### 01--Fractal Analysis of Radiographic Trabecular Bone Texture and Bone Mineral Density: Two Complementary Parameters Related to Osteoporotic Fractures

###### 01--Wavelet-based multifractal formalism to assist in diagnosis in digitized mammograms

###### 09--Fractal and multifractal analysis: A review

### Physics

###### 06--Anisotropic self-affine properties of experimental fracture surface

###### 00--A wavelet-based method for multifractal image analysis. III. Applications to high-resolution satellite images of cloud structure

### Biology

###### 01--Multifractal random walk in copepod behavior

### Climate research

###### 13--The weather and climate: emergent laws and multifractal cascades

### Art investigation

###### 08--Multifractal analysis and authentication of Jackson Pollock paintings,

###### 13--When Van Gogh meets Mandelbrot: Multifractal classification of painting’s texture

###### 14--Pursuing automated classification of historic photographic papers from raking light photomicrographs

# Definition of Multivariate Multifractal spectrum

#### 86--Fractal measures and their singularities: The characterization of strange set

#### 96--Multifractal temperature and flux of temperature variance in fully developed turbulence

#### 90-- Joint multifractal measures: Theory and applications to turbulence

### Remain essentially limited to pairs of time series and are of little relevance for M-uplets of data for M ≫ 2

# Multivariate image

#### 11--Multivariate image analysis: A review with application

# Recent method

## ||The standard estimator for **||**

### A simple linear regression of the sample variance of the logarithm of suitable multiresolution quantities over several analysis scales

###### 93--Log-similarity for turbulent flows?

### This method is appealing for its simplicity and low computational cost

### It has been reported to suffer from poor performance, for images of small size

###### 09--Wavelet leaders and bootstrap for multifractal analysis of images

## ||Approach based on a wavelet scattering transform||

#### 15--Intermittent process analysis with scattering moments

## ||Approaches based parametric model ||

### Paper

##### Maximum likelihood methods

92--Estimation of fractal signals from noisy measurements using wavelets

94--Statistics for Long-Memory Processes

06--Estimation of long-memory time series models: A survey of different likelihood-based methods

08-- A wavelet Whittle estimator of the memory parameter of a nonstationary Gaussian time series

##### Generalized method of moments

07--Higher dimensional multifractal processes: A GMM approach

### Conclusion

##### Their definitions are tied to specific instances of self-similar or multifractal processes

##### Their use of fully parametric models is often too restrictive in practice

## ||Bayesian estimation framework||

### Paper

###### 15-TIP-Bayesian estimation of the multifractality parameter for image texture using a Whittle approximation

###### 15--Bayesian estimation of the multifractality parameter for images via a closed-form Whittle likelihood

### Conclusion

##### It relies on a flexible semi-parametric model for the statistics of the logarithm of wavelet leaders that is generically valid for self-similar and MMC processes

##### Was shown to yield excellent performance

##### However not suited for the analysis of multivariate data since the use of joint priors for collections of parameters leads to inference procedures that are not efficient

# Goal

### Propose the first operational approach for the multifractal analysis of multivariate image

### Propose to conduct the analysis within a multivariate Bayesian model that jointly describes the collection of multifractality parameters c2 associated with the multifractal spectra of each individual data component

##### Refer

15-TIP-Bayesian estimation of the multifractality parameter for image texture using a Whittle approximation

15--Bayesian estimation of the multifractality parameter for images via a closed-form Whittle likelihood

# Main Idea

### Starting from

###### 15-TIP-Bayesian estimation of the multifractality parameter for image texture using a Whittle approximation

###### 15--Bayesian estimation of the multifractality parameter for images via a closed-form Whittle likelihood

### Associated with a likelihood for which **inverse-gamma** distributions are conjugate priors

##### Leading to efficient inference procedures and lending itself well for extension to multivariate data.

##### Three key ingredients

**The Whittle approximation:**   
build a complex Gaussian model for the Fourier coefficients of log-leaders

**A suitable reparametrization**  
the implicit joint constraint on the parameters of the model are then transformed into independent positivity constraints

**Data augmentation extension**  
obtain an augmented distribution whose parameters are easier to estimate

### Use this model in the formulation of a joint Bayesian model for multivariate images

##### Gamma Markov random field (GMRF) priors for the parameters of interest

the prior belief that the multifractality parameter evolves slowly across time or spectral bands (for sequence of images) or throughout space (for image patches)

Induce positive correlation between the multifractality parameters associated with different image components and hence regularize estimation

##### This leads to conditionals for the parameters of the augmented Fourier domain model that are inverse-gamma distributions

##### The inference of the parameters of the associated posterior distribution an be conducted by a Gibbs sampler

# Multifractal Analysis

## ||Holder Exponent||

### Definition

##### The image is said to belong to if

##### The Holder exponent at position is the largest value of such that the inequality holds

##### It is defined as the **Hausdorff dimension** of the sets of positions have identical Holder exponent

### Reference

###### 03--Multifractal processes

###### **04--Wavelet techniques in multifractal analysis**

### Attribute

##### The smaller , the rougher at

##### The larger , the smoother at

##### The goal of multifractal analysis is the estimation of the multifractal spectrum , which provides a global description of the spatial fluctuations of

##### In most applications, the estimation of cannot be based directly on its definition

## ||Multifractal formalism||

### Theory

##### Construct based on multiresolution coefficients, essentially capturing the content of the image around the discrete spatial location k for a given frequency scale

##### Examples are given by increments, wavelet coefficients and wavelet leaders

### Definition

##### The multifractal formalism provides an expansion of the multifractal spectrum of the image ; When

is so-called *log-cumulants*

cannot be positive theoretically

##### Referrence

09-SP-Wavelet leaders and bootstrap for multifractal analysis of image

15--Irregularities and scaling in signal and image processing: Multifractal analysis

### Attributes of

##### The leading order coefficients provide a relevant summary of the multifractal properties of in applications where it would often not be convenient to handle an entire function

##### is the mode of and can be read as a measure for the average smoothness of

##### referred to as the multifractality or intermittency parameter

, for self-similar processes

tied deeply to additive constructions

, for multifractal multiplicative cascade (MMC) process

based on multiplicative constructions and is hence linked to fundamentally different physical principles

##### Reference

03--Multifractal processes

05--Fractal-Based Point Processes

### Estimation of

##### It’s shown that are tied to the quantities

##### In particular

##### The current standard and benchmark estimator for the parameter

is the linear regression of the sample variance

are suitably defined regression weights

##### Limitation

Images of small size and thus image patches cannot be analyzed in practice

Difficult to discriminate between c2 ≡ 0 and values c2 < 0 that are encountered in applications (typically, c2 lies between −0.01 and −0.08).

##### Reference

93-- Log-similarity for turbulent flows?

09-SP-Wavelet leaders and bootstrap for multifractal analysis of image

07-SP- Bootstrap for empirical multifractal analysis

# Wavelet Leader Multifractal Formalism

## ||Reference||

### Multifractal formalisms have been proposed based on increments or wavelet coefficients

###### 09-SP-Wavelet leaders and bootstrap for multifractal analysis of image

###### 07-SP- Bootstrap for empirical multifractal analysis

### A relevant multifractal formalism can be constructed from the wavelet leaders

###### **04--Wavelet techniques in multifractal analysis**

###### 15--Irregularities and scaling in signal and image processing: Multifractal analysis

###### 07-SP- Bootstrap for empirical multifractal analysis

## ||Wavelet Coefficients||

### Wavelet Transform

##### Assume the image is given in form for discrete sample values

##### 1D DWT is defined as

##### 2D DWT filters is defined using tensor product

### Wavelet coefficients

##### At finest scale

##### At the coarser scale

##### are discard

##### Normalize the wavelet coefficients according to the -norm

##### Reference

06--Estimation of long-memory time series models: A survey of different likelihood-based methods

08-- A wavelet Whittle estimator of the memory parameter of a nonstationary Gaussian time series

### Wavelet leaders

##### Denote

##### The union of this cube with its eight neighbors

##### The wavelet leaders are defined as the largest wavelet coefficient magnitude within this neighborhood over all finer scales

##### The Holder exponent is reproduced as follows

### Wavelet leader multifractal formalism (WLMF)

##### Definition

It has been shown above

##### Negative Regularity

The WLMF can be applied to locally bounded images

A large number of real-world images do not satisfy this prerequisite

A practical solution consists of constructing the WLMF using the modified wavelet coefficients

When α is chosen sufficiently large, the WLMF holds

09-SP- Wavelet leaders and bootstrap for multifractal analysis of images

# Bayesian Framework

## ||Modeling of Log-Wavelet Leaders||

### Log-Wavelet Leaders

### 1D signals the distribution of the log-wavelet leaders can be reasonably well approximated by a Gaussian distribution

### The marginal distributions of for 2D images can be also reasonably well approximated by a Gaussian distribution

### Variance-Covariance Model